JOURNAL OF APPROXIMATION THEORY 71, 3–17 (1992)

On a New Characterization of the Classical Orthogonal Polynomials

H. DETTE

Institut für Mathematische Stochastik, Universität Göttingen, Lotzestrasse 13, D-3400 Göttingen, Germany

AND

W. J. STUDDEN*

Department of Statistics, Purdue University, West Lafayette, Indiana 47907-1399, U.S.A.

Communicated by Walter Van Assche

Received March 22, 1991

In this paper we give a new characterization of the classical orthogonal polynomials (Jacobi, Laguerre, and Hermite polynomials) by a special property of the sequences in their recurrence formula. The results also allow an easy derivation of the asymptotic distribution of the zeros of the classical orthogonal polynomials.

1. INTRODUCTION

There is an extensive literature about the mathematical properties of orthogonal polynomials and their applications in various areas. In the wide class of orthogonal systems the classical polynomials are of particular interest: the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, orthogonal (on the interval $[-1, 1]$) with respect to the measure $(1-x)^\alpha(1+x)^\beta\,dx$ $(\alpha, \beta > -1)$; the Laguerre polynomials $L_n^{(\alpha)}(x)$, orthogonal with respect to the measure $x^\alpha e^{-x}\,dx$ (on $[0, \infty), \alpha > -1$); and the Hermite polynomials which are orthogonal (on the real line) with respect to the measure $e^{-x^2}\,dx$. Characterizations of these polynomials are given in [2, 3]. In this paper we present a new characterization of the classical orthogonal polynomials which is based on the sequence from the recurrence formula (Section 3) and allows a very easy derivation of the asymptotic distribution of the zeros of the polynomials (Section 4).

* Research supported in part by National Science Foundation Grant DMS 88-02535.

0021-9045/92 $5.00

2. Preliminaries

Let $\mathscr{X}=(-\infty, \infty)$, $[0, \infty)$, or $[0, 1]$ and μ denote a probability measure on $\mathscr{X}$ with all moments existing. The Stieltjes transform of μ has the continued fraction expansion

$$\text{(a)} \quad \mathscr{X}=[0,1] \qquad \int_0^1 \frac{d\mu(x)}{z-x}=\frac{1|}{|z}-\frac{\zeta_1|}{|1}-\frac{\zeta_2|}{|z}-\cdots,$$

where $\zeta_1=p_1$, $\zeta_j=q_{j-1}p_j$ $(j\geqslant 2)$, $q_j=1-p_j$, $0\leqslant p_j\leqslant 1$;

$$\text{(b)} \quad \mathscr{X}=[0,\infty) \qquad \int_0^\infty \frac{d\mu(x)}{z-x}=\frac{1|}{|z}-\frac{d_1|}{|1}-\frac{d_2|}{|z}-\cdots,$$

where $d_i\geqslant 0$;

$$\text{(c)} \quad \mathscr{X}=(-\infty,\infty) \qquad \int_{-\infty}^\infty \frac{d\mu(x)}{z-x}=\frac{1|}{|z-b_1}-\frac{a_1|}{|z-b_2}-\frac{a_2|}{|z-b_3}-\cdots,$$

where $a_i\geqslant 0$.

The quantities $\{p_i\}_{i\geqslant 1}$, $\{d_i\}_{i\geqslant 1}$, $\{a_i, b_i\}_{i\geqslant 1}$ can be expressed by determinants of the moments of the measure μ (see [6] or [9]). In this sense every probability measure on $[0, 1]$, $[0, \infty)$, $(-\infty, \infty)$ is characterized by the sequence $\{p_i\}_{i\geqslant 1}$, $\{d_i\}_{i\geqslant 1}$, $\{a_i, b_i\}_{i\geqslant 1}$, respectively. If this sequence is finite (i.e., $p_i\in\{0, 1\}$, $d_i=0$, or $a_i=0$ for some i) the corresponding measure has finite support which is given by the zeros of the polynomial in the denominator of its continuous fraction expansion. The following lemma is concerned with the support of a reversed terminating sequence. Its proof is given in the Appendix.

Lemma 2.1. (a) *Probability measures (on $[0, 1]$) corresponding to the sequences $(p_1, ..., p_m, 0)$ and $(p_m, ..., p_1, 0)$ have the same support.*

(b) *Probability measures (on $[0, 1]$) corresponding to the sequences $(p_1, ..., p_m, 1)$ and $(q_m, ..., q_1, 1)$ have the same support.*

(c) *Probability measures (on $[0, \infty)$) corresponding to the sequences $(d_1, ..., d_m, 0)$ and $(d_m, ..., d_1, 0)$ have the same support.*

(d) *Probability measures (on $(-\infty, \infty)$) corresponding to the sequences $\binom{a_1,\ \dots,\ a_m,\ 0}{b_1,\ \dots,\ b_m,\ b_{m+1}}$ and $\binom{a_m,\ \dots,\ a_1,\ 0}{b_{m+1},\ \dots,\ b_2,\ b_1}$ have the same support.*

The following lemma gives the sequences corresponding to classical orthogonal polynomials (see [13]).

LEMMA 2.2. (a) *The corresponding probability measure (on* $[0, 1]$*) of the sequence*

$$p_{2k} = \frac{k}{\alpha + \beta + 2k + 1}, \qquad p_{2k-1} = \frac{\beta + k}{\alpha + \beta + 2k} \qquad (k \geqslant 1)$$

is the "Jacobi" measure with density proportional to $x^{\beta}(1-x)^{\alpha}$ $(\alpha, \beta > -1)$.

(b) *The corresponding probability measure (on* $[0, \infty)$*) of the sequence*

$$d_{2k} = k, \qquad d_{2k-1} = \alpha + k \qquad (k \geqslant 1)$$

is the "Laguerre" measure with density proportional to $x^{\alpha} e^{-x}$ $(\alpha > -1)$.

(c) *The corresponding probability measure (on* $(-\infty, \infty)$*) of the sequence*

$$a_k = \frac{k}{2} \qquad (k \geqslant 1)$$

is the "Hermite" measure with density proportional to e^{-x^2}.

3. Characterization of the Classical Orthogonal Polynomials

Consider a measure μ on $[0, 1]$ with infinite support and corresponding sequence $p_1, p_2, \ldots$ (note that $p_i \in (0, 1)$ because μ has infinite support). Define μ_n $(n \in \mathbb{N})$ as the probability measure (on $[0, 1]$) which corresponds to the "truncated" sequence $p_1, p_2, \ldots, p_{2n-1}, 0$ and μ_n^{R} as the measure which corresponds to the reversed sequence $p_{2n-1}, p_{2n-2}, \ldots, p_1, 0$. Replacing the $\{p_i\}_{i \geqslant 1}$ by $\{d_i\}_{i \geqslant 1}$ we have a similar definition on the half line $[0, \infty)$. For a probability measure on $(-\infty, \infty)$ we define μ_m as the probability measure which corresponds to the truncated sequence $\binom{a_1, \;\ldots,\; a_{n-1}\; 0}{b_1, \;\ldots,\; b_{n-1}\; b_n}$ and μ_n^{R} as the measure corresponding to the "reserved" sequence $\binom{a_{n-1}, \;\ldots,\; a_1, \; 0}{b_n, \;\ldots,\; b_2, \; b_1}$. From the continued fraction expansions given in Section 2 it is obvious that the support of the measure μ_n is given by exactly n points $(n \in \mathbb{N})$ and by Lemma 2.1 it follows that the "reversed" measure μ_n^{R} has the same points as μ_n. In what follows we are interested in measures μ (on $[0, 1]$, $[0, \infty)$, or $(-\infty, \infty)$) for which the truncated and reversed measure μ_n^{R} puts equal masses on its n support points, i.e.,

$$\mu_n^{\mathrm{R}}(\{x\}) = \frac{1}{n} \qquad \forall x \in \operatorname{supp}(\mu_n^{\mathrm{R}}). \tag{3.1}$$

This result has a geometric interpretation which we illustrate for the case $[0, 1]$. The other cases are similar. Let M_{2n} denote the moment space

$$M_{2n} = \left\{ (c_1, ..., c_{2n}) \mid c_i = \int x^i \, d\mu, \ i = 1, ..., 2n \right\}$$

generated by probability measures. For each $(c_1, ..., c_{2n}) \in M_{2n}$ there corresponds a boundary point $(c_1, ..., c_{2n-1}, \underline{c}_{2n})$ for which $\underline{c}_{2n}$ is a minimum. This corresponds to the "lower principal representation" of $c_1, ..., c_{2n-1}$ or to $(p_1, ..., p_{2n-1}, 0)$. Let D_n denote the nth orthogonal polynomial corresponding to $c_1, ..., c_{2n-1}$. The hyperplane supporting M_{2n} at $(c_1, ..., c_{2n-1}, \underline{c}_{2n})$ is determined by D_n^2 and the corresponding face of M_{2n} has extreme points $(x_i, x_i^2, ..., x_i^{2n})$, $i = 1, ..., n$. The measure μ_n^{R} which puts equal masses on its support points can be viewed as the "center" of this face.

The classical orthogonal polynomials can essentially be characterized as the unique polynomials whose corresponding probability measure μ satisfies the condition (3.1) for all $n \in \mathbb{N}$. More precisely we have the following theorem.

THEOREM 3.1. (a) *A probability measure μ on $[0, 1]$ satisfies* (3.1) *for all $n \in \mathbb{N}$ if and only if μ has Jacobi density proportional to $x^\beta(1-x)^\alpha$* $(\alpha, \beta > -1)$.

(b) *A probability measure μ on $[0, \infty)$ satisfies* (3.1) *for all $n \in \mathbb{N}$ if and only if μ has Laguerre density proportional to* $(1/\beta)(x/\beta)^\alpha e^{-x/\beta}$ $(\alpha > -1, \beta > 0)$.

(c) *A probability measure μ on $(-\infty, \infty)$ satisfies* (3.1) *for all $n \in \mathbb{N}$ if and only if μ has the Hermite density proportional to* $\exp[-((x-b)/\sqrt{a})^2]$ $(a > 0, b \in \mathbb{R})$.

Proof. We only give a proof of (a); parts (b) and (c) are proved in the Appendix. In the first step we show that the measure with density proportional to $x^\beta(1-x)^\alpha$ has in fact the property (3.1).

The measures μ_n and μ_n^{R} have the same support by Lemma 2.1. For the calculation of the weights of μ_n^{R} we consider the Stieltjes transform of μ_n^{R}; namely,

$$\Phi(z) = \sum_{i=1}^{n} \frac{\mu_n^{\mathrm{R}}(\{x_i\})}{z - x_i} = \frac{C_n(z)}{D_n(z)}.$$

The demonimator $D_n(z)$ is the Jacobi polynomial $G_n^{(p,q)}(z)$ on $[0, 1]$ with parameters $p = \alpha + \beta + 1$ and $q = \beta + 1$ (see Abramowitz and Stegun [1, p. 782]). We will show that the numerator $C_n(z) = G_{n-1}^{(p,q)}(z)$ with parameters $p=\alpha+\beta+3$ and $q=\beta+2$. In this case we obtain for the weights of μ_n^R at the support points $x_1, x_2, ..., x_n$

$$\begin{aligned}\mu_n^R(\{x_i\}) &= \Phi(z)(z-x_i)|_{z=x_i}\\ &= \frac{C_n(x_i)}{(d/dz)\,D_n(z)|_{z=x_i}} = \frac{G_{n-1}^{(\alpha+\beta+3,\,\beta+2)}(x_i)}{(d/dz)\,G_n^{(\alpha+\beta+1,\,\beta+1)}(z)|_{z=x_i}}\\ &= \frac{1}{n},\end{aligned}$$

where we have used the identity $(d/dz)\,G_n^{(p,q)}(z)=nG_{n-1}^{(p+2,\,q+1)}(z)$.

In order to show that $C_n(z)=G_{n-1}^{(p,\,q)}(z)$ with parameters $p=\alpha+\beta+3$ and $q=\beta+2$ we consider the reversed sequence

$$\tilde{p}_{2i}^{(n)}=p_{2n-2i}=\frac{n-i}{2(n-i)+1+\alpha+\beta},$$

$$\tilde{p}_{2i-1}^{(n)}=p_{2n-2i+1}=\frac{\beta+n-i+1}{2(n-i+1)+\alpha+\beta}$$

$(i=1, ..., n)$ and obtain by an even contraction for the Stieltjes transform of μ_n^R

$$\begin{aligned}\Phi(z) &= \sum_{i=1}^{n}\frac{\mu_n^R(\{x_i\})}{z-x_i}=\int_0^1\frac{\mu_n^R(x)}{z-x}\\ &= \frac{1|}{|z-\tilde{\zeta}_1}-\frac{\tilde{\zeta}_1\tilde{\zeta}_2|}{|z-\tilde{\zeta}_2-\tilde{\zeta}_3}-\frac{\tilde{\zeta}_3\tilde{\zeta}_4|}{|z-\tilde{\zeta}_4-\tilde{\zeta}_5}-\cdots\frac{\tilde{\zeta}_{2n-3}\tilde{\zeta}_{2n-2}|}{|z-\tilde{\zeta}_{2n-2}-\zeta_{2n-1}}=\frac{C_n(z)}{D_n(z)},\end{aligned}$$

where $\tilde{\zeta}_1=\tilde{p}_1^{(n)}$, $\tilde{\zeta}_i=\tilde{p}_i^{(n)}\tilde{q}_{i-1}^{(n)}$ $(i\geqslant 2)$, $D_n(z)=\prod_{i=1}^n(z-x_i)$, and for $k=3, ..., n$

$$C_k(z)=K\begin{pmatrix} & -\tilde{\zeta}_{2n-2k+3}\tilde{\zeta}_{2n-2k+4}\cdots & -\tilde{\zeta}_{2n-3}\tilde{\zeta}_{2n-2} & \\ z-\tilde{\zeta}_{2n-2k+2}-\tilde{\zeta}_{2n-2k+3} & & \cdots & z-\tilde{\zeta}_{2n-2}-\tilde{\zeta}_{2n-1}\end{pmatrix}$$

(see [9] or [14]). Here we have used the usual notation for the continuant K which is also defined in the Appendix. Thus we have for the polynomials $C_n(z)$ the recursive relation

$$C_1(z)=1, \qquad C_2(z)=z-\tilde{\zeta}_{2n-2}-\tilde{\zeta}_{2n-1}=z-\frac{\beta+2}{\alpha+\beta+4},$$

$$C_k(z)=(z-\tilde{\zeta}_{2n-2k+2}-\tilde{\zeta}_{2n-2k+3})\,C_{k-1}(z)-\tilde{\zeta}_{2n-2k+3}\tilde{\zeta}_{2n-2k+4}C_{k-2}(z).$$

Comparing this with the recursive relation of the Jacobi polynomials on $[0, 1]$ with parameters $p=\alpha+\beta+3$, $q=\beta+2$ we find (see Abramowitz and Stegun [1, p. 782])

$$C_k(z)=G_{k-1}^{(\alpha+\beta+3,\beta+2)}(z).$$

For the reverse direction we now show that the Jacobi measure is definitely determined by the condition (3.1). To this end consider an (infinite) measure μ on $[0, 1]$ with corresponding sequence $p_1, p_2, \ldots$ and for $n \in \mathbb{N}$ let μ_n and μ_n^{R} denote the measures corresponding to the sequences $(p_1, \ldots, p_{2n-1}, 0)$ and $(p_{2n-1}, \ldots, p_1, 0)$. The Stieltjes transform of the reversed sequence is given by

$$\Phi(z)=\int_0^1 \frac{d\mu_n^{\mathrm{R}}(x)}{z-x}=\frac{1|}{|z}-\frac{p_{2n-1}|}{|1}-\frac{\gamma_{2n-1}|}{|z}\cdots-\frac{\gamma_2|}{|1}, \tag{3.2}$$

where $\gamma_j=q_jp_{j-1}$ $(j\geqslant 2)$. On the other hand μ_n^{R} puts equal weights on its support points $x_1, \ldots, x_n$ and we obtain

$$\begin{aligned}\Phi(z)=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{z-x_i}=\frac{1}{n}\Bigg[&nz^{n-1}-(n-1)\left(\sum_{i=1}^{n}x_i\right)z^{n-2}\\ &+(n-2)\left(\sum_{i<j}x_ix_j\right)z^{n-3}\cdots\Bigg]\Bigg/\prod_{j=1}^{n}(z-x_j).\end{aligned} \tag{3.3}$$

Because the measures μ_n^{R} and μ_n have the same support (Lemma 2.1) we see from the continued fraction expansion of the Stieltjes transform of the measure μ_n that

$$\prod_{j=1}^{n}(z-x_j)=z^n-\left(\sum_{j=1}^{n}\zeta_j\right)z^{n-1}+\left(\sum_{i=1}^{n}\sum_{j=i+2}^{n}\zeta_i\zeta_j\right)z^{n-2}\cdots$$

which yields

$$\sum_{j=1}^{n}x_j=\sum_{j=1}^{n}\zeta_j, \qquad \sum_{i=1}^{n}\sum_{j=i+1}^{n}x_ix_j=\sum_{i=1}^{n}\sum_{j=i+2}^{n}\zeta_i\zeta_j.$$

By a combination of the equations (3.2) and (3.3) and a comparison of the coefficients of z^{n-2} and z^{n-3} in the polynomials of the numerators we now obtain for all $n\geqslant 2$ the equations

$$\begin{gathered} n\left(\sum_{j=2}^{2n-1}\gamma_j\right)=(n-1)\left(\sum_{j=1}^{2n-1}\zeta_j\right)\\ n\left(\sum_{i=2}^{2n-1}\sum_{j=i+2}^{2n-1}\gamma_i\gamma_j\right)=(n-2)\left(\sum_{i=1}^{2n-1}\sum_{j=i+2}^{2n-1}\zeta_i\zeta_j\right).\end{gathered} \tag{3.4}$$

Observing the identities $\sum_{j=1}^{2n-1} \zeta_j = p_{2n-1} + \sum_{j=2}^{2n-1} \gamma_j$ and $\sum_{i=1}^{2n-1} \sum_{j=i+2}^{2n-1} \zeta_i \zeta_j = \sum_{i=2}^{2n-1} \sum_{j=i+2}^{2n-1} \gamma_i \gamma_j + p_{2n-1} \sum_{j=2}^{2n-2} \gamma_j$ which follow readily from Lemma 2.1, we obtain from (3.4) the equations

$$\begin{aligned} (n-1)\, p_{2n-1} &= \sum_{j=2}^{2n-1} \gamma_j \\ (n-2)\, p_{2n-1} \left(\sum_{j=2}^{2n-2} \gamma_j \right) &= 2 \sum_{i=2}^{2n-1} \sum_{j=i+2}^{2n-1} \gamma_i \gamma_j \qquad \text{for all} \quad n \geqslant 2. \end{aligned} \tag{3.5}$$

We now simplify (3.5) using that the equations must hold for all $n \geqslant 2$ which yields (for $n-1$)

$$\begin{aligned} \sum_{j=2}^{2n-3} \gamma_j &= (n-2)\, p_{2zn-3}, \\ 2 \sum_{i=2}^{2n-3} \sum_{j=i+2}^{2n-3} \gamma_i \gamma_j &= (n-3)\, p_{2n-3} (\gamma_{2n-4} + (n-3)\, p_{2n-5}). \end{aligned}$$

Thus (3.5) reduces to $(n \geqslant 3)$

$$\begin{aligned} &p_{2n-1} [(n-1) + p_{2n-2}] = p_{2n-2} + \gamma_{2n-2} + (n-2)\, p_{2n-3} \\ &(n-2)\, p_{2n-1}\, [(n-1) + p_{2n-2}] \\ &\quad = 2(n-2)\, p_{2n-2} + [2q_{2n-2} + (n-3)][\gamma_{2n-4} + (n-3)\, p_{2n-5}]. \end{aligned} \tag{3.6}$$

We now prove sucessively (for $n \geqslant 2$) that the solutions of the equation (3.5) or equivalently (3.6) are given by

$$p_{2n-1} = \frac{(n-1)\, p_2 + \gamma_2}{(2n-3)\, p_2 + 1} \qquad p_{2n-2} = \frac{(n-1)\, p_2}{(2n-4)\, p_2 + 1} \qquad n = 2, 3, \ldots. \tag{3.7}$$

In the case $n = 2$ we obtain from (3.5) (note that this case gives only one equation for p_3)

$$p_3 = \gamma_2 + (1 - p_3)\, p_2$$

which obviously gives (3.7) for $n = 2$ (the second representation in (3.7) is obvious for $n = 2$). Now assume that the respectation (3.7) holds from 1 to $n-1$ and consider (3.6) for n. By straightforward calculations (using the induction hypothesis) we obtain

$$\gamma_{2n-4} + (n-3)\, p_{2n-5} = (n-2)\, p_{2n-5} \frac{(2n-7)\, p_2 + 2}{(2n-6)\, p_2 + 1}. \tag{3.8}$$

Equating the two equations of (3.6), solving with respect to p_{2n-2}, and using (3.8) it follows that

$$p_{2n-2}\left[1+p_{2n-3}-2p_{2n-5}\frac{(2n-7)p_2+1}{(2n-6)p_2+1}\right]$$
$$=(n-1)\left[p_{2n-3}-p_{2n-5}\frac{(2n-7)p_2+1}{(2n-6)p_2+1}\right]. \tag{3.9}$$

Now observing the representations

$$p_{2n-3}-p_{2n-5}\frac{(2n-7)p_2+1}{(2n-6)p_2+1}=\frac{p_2[(n-3)p_2+(1-\gamma_2)]}{[(2n-5)p_2+1][(2n-6)p_2+1]}$$
$$1-p_{2n-5}\frac{(2n-7)p_2+1}{(2n-6)p_2+1}=\frac{(n-3)p_2+(1-\gamma_2)}{(2n-6)p_2+1}$$

(which follow from the induction hypothesis) we have from (3.9)

$$p_{2n-2}[(n-3)p_2+1-\gamma_2][(2n-5)p_2+1+p_2]$$
$$=(n-1)p_2\,[(n-3)p_2+1-\gamma_2]$$

which reduces to

$$p_{2n-2}=\frac{(n-1)p_2}{(2n-4)p_2+1}.$$

From the first equation given in (3.6) we now obtain (using (3.8)) by straightforward algebra

$$p_{2n-1}=\frac{(n-1)p_2+\gamma_2}{(2n-3)p_2+1}$$

which shows that the solution of the equations (3.6) is given by (3.7). Because every probability measure on $[0, 1]$ which satisfies (3.1) for all $n\in\mathbb{N}$ must also satisfy the equations (3.6), we have shown that the corresponding sequence $p_1, p_2, \ldots$ of such a probability measure on $[0, 1]$ which satisfies (3.1) is determined by

$$p_{2n-1}=\frac{(n-1)p_2+q_2p_1}{(2n-3)p_2+1},\qquad p_{2n-2}=\frac{(n-1)p_2}{(2n-4)p_2+1}\qquad (n\geqslant 2).$$

If we replace the free parameters p_1, p_2 by

$$p_1=\frac{\beta+1}{\alpha+\beta+2},\qquad p_2=\frac{1}{\alpha+\beta+3}\qquad (a, \beta>-1)$$

we obtain

$$p_{2n-2}=\frac{n-1}{2n-1+\alpha+\beta}, \qquad p_{2n-1}=\frac{\beta+n}{2n+\alpha+\beta}$$

and by an application of Lemma 2.2(a) the assertion of the theorem follows.

COROLLARY 3.2. *The Jacobi polynomials can be characterized as the unique orthogonal polynomials on* $[-1, 1]$ *whose corresponding measure satisfies* (3.1).

The Laguerre polynomials can be characterized as the unique (up to a scale factor) orthogonal polynomials on $[0, \infty)$ *whose corresponding measure satisfies* (3.1).

The Hermite polynomials can be characterized as the unique (up to a linear transformation) orthogonal polynomials on $(-\infty, \infty)$ *whose corresponding measure satisfies* (3.1).

Note that in the last corollary we consider the Jacobi polynomials on $[-1, 1]$ while Theorem 3.1 deals with polynomials on $[0, 1]$ for which the derivation of the used equations is easier. By a linear transformation we obtain the desired result on the interval $[-1, 1]$, where the p_i are the same as on the interval $[0, 1]$ (see [10]) and defined by the continued fraction expansion of the Stieltjes transform

$$\int_{-1}^{1} \frac{d\mu(x)}{z-x}=\frac{1|}{|z+1}-\frac{2\zeta_1|}{|1}-\frac{2\zeta_2|}{|z+1}\cdots.$$

We remark that in Theorem 3.1 we required Eq. (3.1) to hold for all $n \in \mathbb{N}$. This is equivalent to requiring that (3.2) equal (3.3) for all n. However, in deriving Eq. (3.4) we only compared the coefficients of z^{n-2} and z^{n-3}. Comparing the remaining coefficients would actually overdetermine our parameters. We conjecture that parts a) and b) are true if we require that (3.1) hold only for $n=1+2^m$, $m \geqslant 1$ and that c) holds if we require (3.1) for $n=2^m$, $m \geqslant 0$.

4. THE ASYMPTOTIC DISTRIBUTION OF THE ZEROS

This section deals with the asymptotic distribution of the zeros of the classical orthogonal polynomials. The results are well known (see [11] for the bounded interval or [8] for an unbounded interval; see also the recent monograph of Van Assche [12]). The proofs either require certain extremal principles from potential theory or are based on the three term recurrence relation and quadrature formulas. An alternative approach was given by Gawronski [5] which uses a continuity theorem for the Stieltjes transform. The results of the previous section allow very simple proofs of the asymptotic distribution of the zeros of classical orthogonal polynomials.

THEOREM 4.1. *Let* $P_n^{(\alpha,\beta)}(x)$ *denote the Jacobi polynomial of degree* n $(\alpha > -1, \beta > -1)$ *and* $N_n^{(\alpha,\beta)}(\xi)$ *the number of zeros of* $P_n^{(\alpha,\beta)}(x)$ *not exceeding* ξ $(\xi \in [-1, 1])$; *then*

$$\lim_{n \to \infty} \frac{1}{n} N_n^{(\alpha,\beta)}(\xi) = \frac{1}{\pi} \int_{-1}^{\xi} \frac{dx}{\sqrt{1-x^2}}.$$

Proof. From Section 3 we know that the corresponding sequence of the Jacobi measure is given by (Lemma 2.2)

$$p_{2n-1} = \frac{\beta+n}{2n+\alpha+\beta}, \qquad p_{2n} = \frac{n}{2n+1+\alpha+\beta}$$

and that the Jacobi polynomials are characterized as the polynomials for which the measure corresponding to the truncated and reversed sequence

$$(p_{2n-1}, ..., p_1, 0) = (\tilde{p}_1^{(n)}, ..., \tilde{p}_{2n-1}^{(n)}, 0)$$

puts equal masses on its support points (namely, the zeros of $P_n^{(\alpha,\beta)}(x)$). From

$$\tilde{p}_{2i}^{(n)} = p_{2n-2i} = \frac{n-i}{2(n-i)+1+\alpha+\beta},$$

$$\tilde{p}_{2i-1}^{(n)} = p_{2n-2i+1} = \frac{\beta+n-i+1}{2(n-i+1)+\alpha+\beta}$$

we have

$$\lim_{n \to \infty} \tilde{p}_{2i}^{(n)} = \lim_{n \to \infty} \tilde{p}_{2i-1}^{(n)} = \tfrac{1}{2}. \tag{4.1}$$

Because the moments of a probability measure are continuous functions of the quantities p_i (see [10]) it follows from (4.1) that the moments of the discrete uniform distribution on the zeros of $P_n^{(\alpha,\beta)}(x)$ converge to the moments of the distribution which corresponds to the sequence $(\frac{1}{2}, \frac{1}{2}, ...)$. This distribution is the arcsine distribution (see for example [7]) and is determined by its moments. It now follows from the well known method of moments of probability theory (see for example Feller [4, p. 263]) that the discrete uniform distribution on the zeros of $P_n^{(\alpha,\beta)}(x)$ converges to the arcsine distribution, because its moments are converging.

THEOREM 4.2. *Let $L_n^{(\alpha)}(x)$ denote the Laguerre polynomial of degree n ($\alpha > -1$) and $N_n^{(\alpha)}(\xi)$ the number of zeros of $L_n^{(\alpha)}(x)$ not exceeding ξ ($\xi \geqslant 0$); then*

$$\lim_{n \to \infty\}} \frac{1}{n} N_n(4n\xi) = \frac{2}{\pi} \int_0^{\xi} x^{-1/2}(1-x)^{1/2}\, dx \qquad (0 \leqslant \xi \leqslant 1).$$

Proof. Let $x_1, ..., x_n$ denote the zeros of $L_n^{(\alpha)}(x)$. From Theorem 3.1 and Corollary 3.2 we see that the corresponding sequence of the discrete uniform distribution on the set $\{x_i/4n\}_{i=1}^{n}$ is given by

$$(d_1^{(n)}, ..., d_{2n-1}^{(n)}, 0),$$

where

$$d_{2i}^{(n)} = (n-i)/4n \qquad d_{2i-1}^{(n)} = (\alpha + n - i + 1)/4n \qquad (i = 1, ..., n)$$

and we obtain as $n \to \infty$

$$\lim_{n \to \infty} d_{2i}^{(n)} = \tfrac{1}{4} \qquad \lim_{n \to \infty} d_{2i-1}^{(n)} = \tfrac{1}{4}.$$

The only distribution (on $[0, 1]$) with corresponding sequence $(\frac{1}{4}, \frac{1}{4}, ...)$ is the distribution with density $x^{-1/2}(1-x)^{1/2}$ ($\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$ in Lemma 2.2a) and the assertion of the theorem now follows by similar arguments given in the proof of Theorem 4.1.

The next theorem is proved in the same way as Theorems 4.1 and 4.2 and its proof is therefore omitted.

THEOREM 4.3. *Let $H_n(x)$ denote the Hermite polynomial of degree n and $N_n(\xi)$ the number of zeros of $H_n(x)$ not exceeding ξ ($\xi \in \mathbb{R}$); then*

$$\lim_{n \to \infty} \frac{1}{n} N_n(\sqrt{2n}\xi) = \frac{2}{\pi} \int_{\pi}^{\xi} \sqrt{1-x^2}\, dx \qquad (-1 \leqslant \xi \leqslant 1).$$

5. APPENDIX

Proof of Lemma 2.1. We only give a proof of (a); all other cases are treated similarly. The Stieltjes transforms of the measures μ_n and μ_n^{R} corresponding to the sequences $(p_1, p_2, ..., p_m, 0)$ and $(p_m, p_{m-1}, ..., p_1, 0)$ are given by

$$\int_0^1 \frac{d\mu_n(x)}{z-x} = \frac{1|}{|z} - \frac{\zeta_1|}{|1} - \frac{\zeta_2|}{|z} - \cdots - \frac{\zeta_m|}{|\tau_m}$$

$$\int_0^1 \frac{d\mu_n^{\mathrm{R}}(x)}{z-x} = \frac{1|}{|z} - \frac{p_m|}{|1} - \frac{\gamma_m|}{|z} - \cdots - \frac{\gamma_2}{|\tau_m},$$

where $\zeta_1 = p_1$, $\zeta_j = p_j(1-p_{j-1})$, $\gamma_j = (1-p_j)\,p_{j-1}$, and τ_m is 1 or z corresponding to the case m odd or even. The support points of μ_n and μ_n^{R} are given by the zeros of the polynomials in the denominator (see [14]).

$$P_m(z) = K\begin{pmatrix} & -\zeta_1 & & -\zeta_2 & & \cdots & -\zeta_m & \\ z & & 1 & & z & \cdots & & \tau_m \end{pmatrix}$$

$$= \begin{vmatrix} z & -1 & 0 & \cdots & 0 \\ -\zeta_1 & 1 & -1 & & \vdots \\ 0 & -\zeta_2 & z & & \vdots \\ \vdots & & \ddots & & 0 \\ \vdots & & & & -1 \\ 0 & \cdots & 0 & -\zeta_m & \tau_m \end{vmatrix}$$

$$Q_m(z) = K\begin{pmatrix} & -p_m & & -\gamma_m & & \cdots & -\gamma_2 & \\ z & & 1 & & z & \cdots & & \tau_m \end{pmatrix}$$

$$= \begin{vmatrix} z & -1 & 0 & \cdots & 0 \\ -p_m & 1 & -1 & & \vdots \\ 0 & -\gamma_m & z & & \vdots \\ \vdots & & \ddots & & 0 \\ \vdots & & & & -1 \\ 0 & \cdots & 0 & -\gamma_2 & \tau_m \end{vmatrix}$$

We now prove by induction that the polynomials $P_m(x)$ and $Q_m(x)$ are the same. For $m=1$ this is obvious and for $m=2$ we obtain

$$\begin{aligned} P_2(z) &= z^2 - z(\zeta_1+\zeta_2) = z^2 - z(p_1 + q_1 p_2) = z^2 - z(p_2 + q_2 p_1) \\ &= z^2 - z(p_2 + \gamma_1) = Q_2(z). \end{aligned}$$

For the step from m to $m+1$ we have from the induction hypothesis (for $m-1$ and $m-2$) and by an expansion in the last row

$$\begin{aligned} P_{m+1}(z) &= \tau_{m+1} K\begin{pmatrix} & -\zeta_1 & & \cdots & -\zeta_m & \\ z & & 1 & \cdots & & \tau_m \end{pmatrix} \\ &\quad - \zeta_{m+1} K\begin{pmatrix} & -\zeta_1 & & \cdots & -\zeta_{m-1} & \\ z & & 1 & \cdots & & \tau_{m-1} \end{pmatrix} \\ &= \tau_{m+1} K\begin{pmatrix} & -p_m & & -\gamma_m & & \cdots & -\gamma_2 & \\ z & & 1 & & z & \cdots & & \tau_m \end{pmatrix} \\ &\quad - \zeta_{m+1} K\begin{pmatrix} & -p_{m-1} & & -\gamma_{m-1} & & \cdots & -\gamma_2 & \\ z & & 1 & & z & \cdots & & \tau_{m-1} \end{pmatrix}. \end{aligned}$$

From the identity $p_k = \gamma_{k+1} + p_k p_{k+1}$ $(k = m, m-1)$ we obtain

$$
\begin{aligned}
P_{m+1}(z) = {}& \tau_{m+1} K\begin{pmatrix} & -\gamma_{m+1} & & -\gamma_m & & \cdots & -\gamma_2 & \\ z & & 1 & & z & \cdots & & \tau_m \end{pmatrix} \\
& - \zeta_{m+1} K\begin{pmatrix} & -\gamma_m & & -\gamma_{m-1} & & \cdots & -\gamma_2 & \\ z & & 1 & & z & \cdots & & \tau_{m-1} \end{pmatrix} \\
& - p_m p_{m+1}\left[\tau_{m+1} K\begin{pmatrix} & -\gamma_{m-1} & & & -\gamma_2 & \\ z & & 1 & \cdots & & \tau_m \end{pmatrix}\right. \\
& \left. - \gamma_m K\begin{pmatrix} & -\gamma_{m-2} & & \cdots & -\gamma_2 & \\ z & & 1 & \cdots & & \tau_{m-1} \end{pmatrix}\right] \\
= {}& \tau_{m+1} K\begin{pmatrix} & -\gamma_{m+1} & & \cdots & -\gamma_2 & \\ z & & 1 & \cdots & & \tau_m \end{pmatrix} \\
& - \zeta_{m+1}\tau_{m+1} K\begin{pmatrix} & -\gamma_m & & \cdots & -\gamma_2 & \\ 1 & & z & \cdots & & \tau_m \end{pmatrix} \\
& - p_m p_{m+1}\left[\tau_{m+1} K\begin{pmatrix} & -\gamma_{m-1} & & \cdots & -\gamma_2 & \\ z & & 1 & & & \tau_m \end{pmatrix}\right. \\
& \left. - \tau_{m+1}\gamma_m K\begin{pmatrix} & -\gamma_{m-2} & & \cdots & -\gamma_2 & \\ 1 & & z & \cdots & & \tau_m \end{pmatrix}\right] \\
= {}& \tau_{m+1}\left[K\begin{pmatrix} & -\zeta_{m+1} & & -\gamma_{m+1} & & \cdots & -\gamma_2 & \\ 1 & & z & & 1 & \cdots & & \tau_m \end{pmatrix}\right. \\
& \left. - p_m p_{m+1} K\begin{pmatrix} & -\gamma_m & & \cdots & -\gamma_2 & \\ 1 & & z & \cdots & & \tau_m \end{pmatrix}\right] \\
= {}& K\begin{pmatrix} & -p_{m+1} & & -\gamma_{m+1} & & \cdots & -\gamma_2 & \\ z & & 1 & & z & \cdots & & \tau_{m+1} \end{pmatrix} = Q_{m+1}(z).
\end{aligned}
$$

(Note that we have used the identity

$$
\begin{aligned}
& K\begin{pmatrix} & -\gamma_m & & -\gamma_{m-1} & & \cdots & -\gamma_2 & \\ z & & 1 & & z & \cdots & & \tau_{m-1} \end{pmatrix} \\
& \quad = \tau_{m+1} K\begin{pmatrix} & -\gamma_m & & -\gamma_{m-1} & & & -\gamma_2 & \\ 1 & & z & & 1 & \cdots & & \tau_m \end{pmatrix}
\end{aligned}
$$

which can also be proved by induction and straightforward calculations.)

Proof of Theorem 3.1 (b) *and* (a). The proofs that the Laquerre and Hermite densities satisfy (3.1) are similar to the Jacobi case and are omitted. The proofs that these densities are determined by (3.1) are given below.

Proof of part (b). By a similar reasoning as in the proof of part (a) we obtain the equations

$$\left.\begin{aligned} nd_{2n-1} &= \sum_{i=1}^{2n-1} d_i \\ (n-1)\left(\sum_{i=1}^{2n-3} d_i\right) d_{2n-1} &= 2\sum_{i=1}^{2n-2}\sum_{j=i+2}^{2n-2} d_i d_j \end{aligned}\right\} \quad \text{for all} \quad n \geqslant 2 \tag{5.1}$$

which reduce to $(\sum_{i=1}^{2n-3} d_i = (n-1)\, d_{2n-3})$

$$\begin{aligned} (n-1)\, d_{2n-1} &= d_{2n-2} + (n-1)\, d_{2n-3} \\ (n-2)(n-1)\, d_{2n-1}\, d_{2n-3} &= 2\sum_{i=1}^{2n-2}\sum_{j=i+2}^{2n-2} d_i\, d_j. \end{aligned} \tag{5.2}$$

For the second equation in (5.2) we obtain

$$\begin{aligned} &(n-2)(n-1)\, d_{2n-1}\, d_{2n-3} \\ &\quad = 2\left[d_{2n-2}\sum_{i=1}^{2n-4} d_i + d_{2n-3}\sum_{i=1}^{2n-5} d_i + \sum_{i=1}^{2n-4}\sum_{j=i+2}^{2n-4} d_i\, d_j \right] \\ &\quad = 2(n-2)\, d_{2n-2}\, d_{2n-3} + 2(n-2)\, d_{2n-3}\, d_{2n-5} \\ &\qquad + (n-3)(n-2)\, d_{2n-3}\, d_{2n-5} \end{aligned}$$

and thus (5.2) is equivalent to

$$\left.\begin{aligned} (n-1)\, d_{2n-1} &= d_{2n-2} + (n-1)\, d_{2n-3} \\ (n-1)\, d_{2n-1} &= 2d_{2n-2} + (n-1)\, d_{2n-5} \end{aligned}\right. \quad \text{for all } n \geqslant 2. \tag{5.3}$$

By a straightforward calculation it can now be shown that the solution of the equations (5.3) is given by

$$d_{2n-2} = (n-1)\, d_2, \qquad d_{2n-1} = d_1 + (n-1)\, d_2$$

and part (b) of Theorem 3.1 now follows replacing d_1 and d_2 by the parameters of the density proportional to $1/\beta(x/\beta)^{\alpha} e^{-x/\beta}$ $(\alpha > -1 \quad \beta > 0)$, which yields $d_1 = (1+\alpha)\,\beta$ and $d_2 = \beta$.

Proof of part (c). In the same way as in part (a) we obtain the equations (for all $n \geqslant 1$)

$$\begin{aligned} n\sum_{i=1}^{n+1} b_i &= (n+1)\sum_{i=1}^{n} b_i \\ (n-1)\left[\sum_{i=1}^{n+1}\sum_{j=i+1}^{n+1} b_i b_j - \sum_{i=1}^{n} a_i\right] &= (n+1)\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} b_i b_j - \sum_{i=1}^{n-1} a_i\right]. \end{aligned} \tag{5.4}$$

From the first equation we get immediately that $b_1 = b_2 = b_3 \ldots$ and we let without loss of generality $b_1 = b_2 = \cdots = 0$ (a nonzero b only causes a shift of the distribution). But in this case the second equation of (5.4) reduces to

$$a_n = \frac{2}{n-1} \sum_{i=1}^{n-1} a_i \qquad (n \geqslant 2)$$

which has the solution $a_n = na_1$ for all $n \in \mathbb{N}$, and Theorem 3.1(c) is proved by an application of Lemma 2.1 and a linear transformation.

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